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On the travelling wave instability caused by moving planar heat sources: the rigid problem

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Abstract. This analysis deals with the convective travelling wave instability appearing in a fluid medium at rest and contained between two horizontal rigid plates, subjected to the same sinusoidal temperature distribution, moving at a uniform speed and in the same direction. The temperature distribution is caused by travelling planar heat sources with a time harmonic output. A three-dimensional coordinate system is used and the small parameter ϵ in this problem represents the ratio between the buoyancy and inertial forces. For a finite, yet small ϵ, asymptotic expansions are assumed for the velocity, pressure, temperature and the Reynolds number. The mean motion generated by the Reynolds stresses is calculated separately. By keeping the Prandtl number fixed and by using long length and time scales, successive linearized perturbation equations are considered. Two successive amplitude equations are analyzed and their solution yields the mathematical form of these travelling waves, their group velocity and the elevation above the critical Reynolds number.

1. Introduction

Several papers have been written on the convective two-dimensional motion of a fluid due to a moving heat source near or at the boundary. Fultz *et al.* [1] and Stern [2] described experiments in which a flame is rotated around the outside bottom rim of a cylindrical vessel filled with water. They found that during the development of the motion from rest, the fluid acquired a net angular momentum in the sense opposite to the motion of the flame. Stern [2] examined a two-dimensional model to investigate whether a travelling heat source with a time harmonic output could indeed impart net momentum to the fluid contained between two horizontal plates. He assumed that each plate was subjected to the same sinusoidal temperature distribution, moving with a unifonn speed and in the same direction. He concentrated on the case when the depth of the fluid is small compared with the horizontal scale of the motion. Davey [3] performed some analytical calculations for the motion due to a moving source of heat, and his results for the mean flow generated by the Reynolds stresses lead to the conclusion that at all frequencies the net mean momentum is in the opposite direction to that of the thermal field. The motivation for these studies is connected with Halley's theory for the atmosphere's general circulation and for the wind angular momentum produced by the westward progressive heating of the earth by the sun.

The mixed-convection problem of a weakly buoyant plume in the presence of an ambient current has been studied by Afzal [4] and Wesseling [5]. Afzal [4] considered a stationary two-dimensional line heat source placed in an on-coming vertical stream, while Wesseling [5] dealt with the buoyant plume induced by a point source in a free stream directed at an

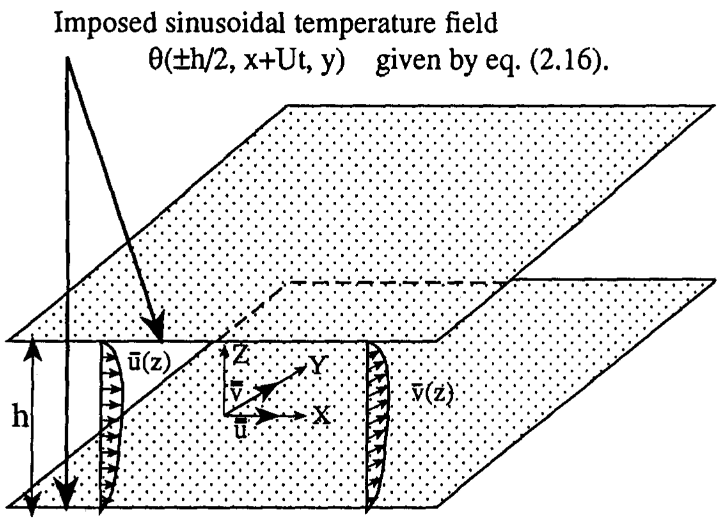

Fig. 1. Moving thermal forcing between two rigid walls.

arbitrary angle with respect to the vertical. The last two mentioned papers deal with a heat source immersed in an infinite expanse of fluid, whereas the work of Miloh and Yahalom [6] considers the influence of both a rigid wall or a free-surface on the plume characteristics.

The present work is connected with a three-dimensional extension of the analyses of Fultz [1], Stern [2] and Davey [3] for more realistic channel flows configurations. It deals with the very fundamental physical question whether a controlled heating of the boundaries of a 3-D domain bounding a quiescent flow can indeed generate a mean momentum flux produced by the Reynolds stresses in a prescribed direction. In addition, it presents some newly derived stability features of the induced convective flow, such as the travelling wave instability and the change in the critical Reynolds number.

2. Basic analysis

In this work we consider the uniform motion of a planar system of heat sources in between and parallel to two rigid plane boundaries in a medium which is otherwise at rest. The fluid is viscous and the flow is three-dimensional. The x and y axes are taken to be horizontal at the mid-depth of the fluid in a channel of depth h (see Figure 1) and the z- axis is taken as the upward normal to the plates. The mean motion, arising from *non-linear* interactions, is defined to be $\bar{u}(z)$ and $\bar{v}(z)$. No basic uniform motion is present and clearly the linearized solution does not render any motion. The mean pressure is $\bar{p}(z)$, and u', v', w', p' and ρ' denote the horizontal and vertical velocity fluctuations, the pressure and density fluctuations, respectively, which arise from the sinusoidal temperature perturbation applied on the boundaries, as explained in the sequel. The mean density of the fluid is ρ_0, the kinematic viscosity is ν and g is the downward vertical component of gravity.

The main object of this paper is to determine whether an imposed time harmonic thermal field on the boundary can produce a horizontal *mean* motion of the fluid and, if this is the case, to find its average horizontal momentum. The Navier–Stokes and the continuity equations are written below as follows:

$$\frac{\partial u'}{\partial t} + (\bar{u} + u')\frac{\partial u'}{\partial x} + (\bar{v} + v')\frac{\partial u'}{\partial y} + w'\frac{\partial}{\partial z}(\bar{u} + u')$$

$$= -\frac{1}{\rho_o}\frac{\partial p'}{\partial x} + \nu(\frac{\partial^2 \bar{u}}{\partial z^2} + \nabla^2 u') \tag{2.1}$$

$$\frac{\partial v'}{\partial t} + (\bar{u} + u')\frac{\partial v'}{\partial x} + (\bar{v} + v')\frac{\partial v'}{\partial y} + w'\frac{\partial}{\partial z}(\bar{v} + v')$$
$$= -\frac{1}{\rho_o}\frac{\partial p'}{\partial y} + \nu(\frac{\partial^2 \bar{v}}{\partial z^2} + \nabla^2 v') \tag{2.2}$$

$$\frac{\partial w'}{\partial t} + (\bar{u} + u')\frac{\partial w'}{\partial x} + (\bar{v} + v')\frac{\partial w'}{\partial y} + w'\frac{\partial w'}{\partial z}$$
$$= -\frac{1}{\rho_o}\left(\frac{\partial p'}{\partial z} + \rho' g\right) + \nu\nabla^2 w' \tag{2.3}$$

$$\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z} = 0. \tag{2.4}$$

The pressure term p' can be eliminated by differentiating the linearized forms of (2.1) and (2.3) with respect to z and x, respectively, and subtracting the two. Differentiating the resulting equation with respect to x, yields:

$$\frac{\partial}{\partial x}\left(\frac{\partial}{\partial t} - \nu\nabla^2\right)\left(\frac{\partial u'}{\partial z} - \frac{\partial w'}{\partial x}\right) = \frac{g}{\rho_o}\frac{\partial^2 \rho'}{\partial x^2}. \tag{2.5}$$

Similarly, by eliminating p' between the linearized forms of (2.1) and (2.2) we have:

$$\frac{\partial}{\partial y}\left(\frac{\partial}{\partial t} - \nu\nabla^2\right)\left(\frac{\partial v'}{\partial z} - \frac{\partial w'}{\partial y}\right) = \frac{g}{\rho_o}\frac{\partial^2 \rho'}{\partial y^2}. \tag{2.6}$$

where the boundary conditions for these equations are $u' = v' = w' = 0$ at $z = \pm h/2$.

Use of the continuity equation (2.4) leads to a linearized perturbation equation in w' and ρ':

$$\left(\frac{\partial}{\partial t} - \nu\nabla^2\right)\nabla^2 w' = -\frac{g}{\rho_o}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\rho'. \tag{2.7}$$

The energy equation is usually given in the following form:

$$\rho c_p\left[\frac{\partial \theta}{\partial t} + (u\cdot\nabla)\theta\right] = k\nabla^2\theta \tag{2.8}$$

where $\theta(x, y, z) = \bar{\theta}(z) + \theta'(x, y, z)$ represents the temperature. The mean and perturbation temperatures are denoted by $\bar{\theta}$ and θ', respectively.

If we consider the linearized version of (2.8) and use dimensionless quantities by the same symbols as the corresponding dimensional ones, we get a system of linear differential equations for the perturbed variables, as follows:

$$\left(\frac{\partial}{\partial t} - \frac{1}{2\lambda}\nabla^2\right)\nabla^2 w' = \epsilon\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\theta' \tag{2.9}$$

$$i\alpha_x\left(\frac{\partial}{\partial t}-\frac{1}{2\lambda}\nabla^2\right)\left(\frac{\partial u'}{\partial z}-i\alpha_x w'\right)=-\epsilon\frac{\partial^2\theta'}{\partial x^2} \tag{2.10}$$

$$i\alpha_y\left(\frac{\partial}{\partial t}-\frac{1}{2\lambda}\nabla^2\right)\left(\frac{\partial v'}{\partial z}-i\alpha_y w'\right)=-\epsilon\frac{\partial^2\theta'}{\partial y^2} \tag{2.11}$$

$$\left(\frac{\partial}{\partial t}-\frac{1}{2\lambda\sigma}\nabla^2\right)\theta'=0 \tag{2.12}$$

where proper account has been taken of the proportionality of ρ' and θ' as indicated by the equation of state. Furthermore, $\lambda = Uh/2\nu$ denotes the Reynolds number, h represents the distance between the two rigid walls and is used here as the length scale, U is the velocity scale, h/U is the time scale, k is the thermal conductivity, $\bar{\beta}$ is the thermal expansion coefficient, μ is the dynamic viscosity, c_p is the specific heat for constant pressure coefficient, $\tilde{\theta}$ is some reference temperature, $\sigma = \mu c_p/k$ is the Prandtl number and $\epsilon = \bar{\beta}gh\tilde{\theta}/U^2$, which is a measure of the ratio between buoyancy and inertial forces, is considered a small parameter in the following analysis. This parameter may also be expressed as $\epsilon = \bar{\beta}\tilde{\theta}/\mathrm{Fr}$, where $\mathrm{Fr} = U^2/gh$ is the Froude number. Finally, α_x and α_y are the corresponding wave numbers in the following Fourier expansions:

$$u' = e^{i\alpha_x(x+Ut)+i\alpha_y y}u_1(z) + e^{-i\alpha_x(x+Ut)-i\alpha_y y}u_1^*(z) \tag{2.13}$$

$$v' = e^{i\alpha_x(x+Ut)+i\alpha_y y}v_1(z) + e^{-i\alpha_x(x+Ut)-i\alpha_y y}v_1^*(z) \tag{2.14}$$

$$w' = e^{i\alpha_x(x+Ut)+i\alpha_y y}w_1(z) + e^{-i\alpha_x(x+Ut)-i\alpha_y y}w_1^*(z) \tag{2.15}$$

and $\alpha = (\alpha_x^2+\alpha_y^2)^{1/2}$ is the wave number of the disturbance. In addition it is assumed that:

$$\begin{aligned}\theta(x,y,z) &= \bar{\theta}(z) + \theta'(x,y,z)\\ &= \bar{\theta}(z) + e^{i\alpha_x(x+Ut)+i\alpha_y y}\theta_1(z) + e^{-i\alpha_x(x+Ut)-i\alpha_y y}\theta_1^*(z)\end{aligned} \tag{2.16}$$

where $\bar{\theta}$ is the mean temperature with respect to x and y. This condition for θ' is imposed in order to have a realistic planar temperature distribution at the boundary. The temperature field is assumed to move with a constant velocity in the x-direction. The longitudinal wave number is α_x and the transverse one is α_y. Letting $\alpha_y = 0$ renders a two-dimensional problem similar to the one formulated by Stern and Davey [3]. Imposing boundary conditions on the side walls will imply integer values for α_y in terms of the channel width. In this problem the values of $\bar{\theta}$ at the upper and lower boundaries are assumed (as in Davey and Stern) to be identical constants. Let us choose the origin of a Cartesian coordinate system half way between the two rigid plates, such that $\bar{\theta}(\mp h/2) = \tilde{\theta} = \text{const.}$

Substitution of (2.13)–(2.16) in the system of equations (2.9)–(2.12) yields a system of differential equations for u_1, v_1, w_1 and θ_1, as follows ($\mathrm{D} = \frac{\mathrm{d}}{\mathrm{d}z}$);

$$(D^2-\alpha^2)(D^2-\alpha_x^2-2i\alpha_x U\lambda)w_1 = 2\lambda\epsilon\alpha^2\theta_1 \tag{2.17}$$

$$(D^2-\alpha^2-2i\alpha_x U\lambda)(Du_1-i\alpha_x w_1) = 2i\lambda\alpha_x\epsilon\theta_1 \tag{2.18}$$

$$(D^2-\alpha^2-2i\alpha_x U\lambda)(Dv_1-i\alpha_y w_1) = 2i\lambda\alpha_y\epsilon\theta_1 \tag{2.19}$$

$$(D^2 - \alpha^2 - 2i\alpha_x U\lambda\sigma)\theta_1 = 0. \tag{2.20}$$

Following Stern [2] attention is confined here to cases where the depth of the fluid is small compared with the wavelength of the thermal field in both x and y directions, i.e., ($\alpha^2 h^2 << 4\pi^2$). This is done only to ease the algebra involved but does not affect qualitatively the outcome. In this way, and using the shallow layer approximation, we obtain:

$$D^2(D^2 - 2i\alpha_x U\lambda)w_1 = 2\lambda\epsilon\alpha^2\theta_1 \tag{2.21}$$

$$D(D^2 - 2i\alpha_x U\lambda)u_1 = 2i\lambda\alpha_x\epsilon\theta_1 \tag{2.22}$$

$$D(D^2 - 2i\alpha_x U\lambda)v_1 = 2i\lambda\alpha_y\epsilon\theta_1 \tag{2.23}$$

$$(D^2 - 2i\alpha_x U\lambda\sigma)\theta_1 = 0 \tag{2.24}$$

where the variable w_1 has first been eliminated from (2.18) and (2.19) and then the system was subject to Stern's approximation.

Before continuing to the next stage the definitions will be completed with

$$\Omega = 2\alpha_x U\lambda\sigma, \quad \bar{\alpha}^2 = \frac{i\Omega}{\sigma}, \quad s^2 = i\Omega \tag{2.25}$$

The quantity Ω measures the ratio of the time scale for heat to diffuse through a distance h, to the time scale imposed by the moving thermal field and thus can be regarded as a frequency parameter.

In this three-dimensional problem, like the two-dimensional version considered by Stern in the past, we assume that the fluid layer is vertically bounded by two rigid horizontal plates.

From (2.24) and (2.25) the solution for θ_1 with the boundary conditions $\theta_1 = \tilde{\theta}$ at $z = \pm 1/2$ is given by:

$$\frac{\theta_1(z)}{\tilde{\theta}} = \frac{\cosh sz}{\cosh \frac{1}{2s}} \tag{2.26}$$

By combining the equation of state with (2.26), (2.21) and (2.25), it may be written as:

$$F^{IV} - \alpha^{-2}F^{II} = \bar{\alpha}^{-2}s^2\frac{\cosh sz}{\cosh^{s/2}} \tag{2.27}$$

where $F(z)$ has been defined by the expression:

$$w_1(z) = -\frac{\epsilon\bar{\theta}(\alpha/\alpha_x)^2}{2U^2\lambda\sigma}F(z) \tag{2.28}$$

and the dashes denote differentiation with respect to z. The boundary conditions $u' = v' = w' = 0$ at the walls are translated to:

$$F = F' = 0 \quad \text{at} \quad z = \pm 1/2 \tag{2.29}$$

The solution of (2.27) and (2.29) is given analytically by:

$$(\sigma - 1)F = \frac{s\tanh^{s/2}}{\bar{\alpha}\tanh^{\bar{\alpha}/2}}\left[1 - \frac{\cosh\bar{\alpha}z}{\cosh^{\bar{\alpha}/2}}\right] - \left[1 - \frac{\cosh sz}{\cosh^{s/2}}\right]. \tag{2.30}$$

The value of F for $\sigma = 1$ can be found from (2.30) by using l'Hôpital's rule.

$$\lim F_{\sigma\to 1} = \frac{1}{2}\left(1 - \frac{\cosh\bar{\alpha}z}{\cosh^{\bar{\alpha}/2}}\right)\left(1 + \frac{\bar{\alpha}}{\sinh\bar{\alpha}}\right) + \frac{\bar{\alpha}\cosh\bar{\alpha}z}{2\cosh^{\bar{\alpha}/2}} \cdot \cdot \left(z\tanh\bar{\alpha}z - \frac{1}{2}\tanh^{\bar{\alpha}/2}\right) \tag{2.31}$$

The mean motion is evaluated by extracting the mean part of (2.1) and (2.2) over one wavelength. The mean motion generated by the Reynolds stress, expressed in non-dimensional form, satisfies (see Hinze [7]) a pair of two differential equations:

$$\frac{d^{2-}\bar{u}}{dz^2} = 2\lambda\frac{d}{dz}(\overline{u'w'}) \tag{2.32}$$

$$\frac{d^2\bar{v}}{dz^2} = 2\lambda\frac{d}{dz}(\overline{v'w'}) \tag{2.33}$$

According to the last two equations and (2.28):

$$\frac{u_1}{\alpha_x} = \frac{v_1}{\alpha_y} = \frac{-i\epsilon\tilde{\theta}}{2\alpha_x U^2\lambda\sigma}F'(z). \tag{2.34}$$

Integrating by parts to remove the double integral and since $F(z)$ is an even function, we determine the average mean velocities $\bar{\bar{u}}$ and $\bar{\bar{v}}$ of the fluid:

$$\frac{\bar{\bar{u}}}{\alpha_x} = \frac{\bar{\bar{v}}}{\alpha_y} = -\frac{1}{2\lambda}\left(\frac{\epsilon\tilde{\theta}}{\alpha_x U^2\sigma}\right)^2 \operatorname{Im}\left[\int_0^{1/2} zF\tilde{F}'dz\right] \tag{2.35}$$

where it should be noted that the average mean velocities are of the order of the square in the fluctuations velocity, and where a second overbar denotes a mean value with respect to z. Im denotes the imaginary part and $\tilde{F}$ is the complex conjugate of F. For large and small values of λ and σ some asymptotic values are found, as follows:

$$\frac{\bar{\bar{u}}}{\alpha_x} = \frac{\bar{\bar{v}}}{\alpha_y} = \frac{1}{2\lambda(1+\sigma)}\left(\frac{\epsilon\tilde{\theta}}{\alpha_x U^2\sigma}\right)^2 \cdot \cdot \left[1 - \frac{3\sigma^2 + \sigma^{3/2} + 10\sigma + \sigma^{1/2} + 3}{\sqrt{2\Omega}(\sigma^{1/2}+1)(\sigma+1)} + O(\Omega^{-1})\right] \quad (\Omega \text{ large}) \tag{2.36}$$

$$\frac{\bar{\bar{u}}}{\alpha_x} = \frac{\bar{\bar{v}}}{\alpha_y} = \frac{2(1+\sigma)}{12!\sigma^3\lambda}\left(\frac{\epsilon\tilde{\theta}}{\alpha_x U^2\sigma}\right)[\Omega^5 + O(\Omega^7)] \quad (\Omega \text{ small}) \tag{2.37}$$

corresponding to each case of Ω large and Ω small. By taking $\alpha_y \equiv 0$, it is possible to retrieve Davey's [3] results in non-dimensional form.

3. Modulation analysis

In the governing equations (2.5)–(2.6) we take an expansion in powers of ϵ for θ_1, w_1 and the Reynolds number λ, as follows:

$$\begin{aligned} \theta_1 &= \theta_{10} + \epsilon\theta_{11} + \epsilon^2\theta_{12} + \cdots \\ w_1 &= \epsilon w_{11} + \epsilon^2 w_{12} + \epsilon^3 w_{13} + \cdots \\ \lambda &= \lambda_o + \epsilon\lambda_1 + \epsilon^2\lambda_2 + \cdots . \end{aligned} \tag{3.1}$$

By now introducing the long X, Y and time τ scales,

$$x = \epsilon^{-1}X, \quad y = \epsilon^{-1}Y, \quad t = \epsilon^{-1}\tau \tag{3.2}$$

we define

$$\theta_{10} = B(X,Y,\tau)g_{10}, \qquad w_{11} = B(X,Y,\tau)f_{11} \tag{3.3}$$

where $B(X,Y,\tau)$ is an amplitude function of the long space and time variables to be determined from the perturbation equations. If f_{1R}, f_{1I} and g_{0R}, g_{0I} are the real and imaginary parts of f_{11} and g_{10}, respectively, then the $O(\epsilon)$ system is as follows:

$$M^2 f_{1R} + 2\alpha_x U\lambda_o M f_{1I} - 2\lambda_o\alpha^2 g_{0R} = 0 \tag{3.4}$$

$$M^2 f_{1I} - 2\alpha_x U\lambda_o M f_{1R} - 2\lambda_o\alpha^2 g_{0I} = 0 \tag{3.5}$$

$$M g_{0R} + 2\alpha_x U\lambda_o\sigma g_{0I} = 0 \tag{3.6}$$

$$M g_{0I} - 2\alpha_x U\lambda_o\sigma g_{0R} = 0 \tag{3.7}$$

where M is a differential operator defined by $M = D^2 - \alpha^2$.

3.1. ADJOINT SYSTEM

In order to find a solution for the subsequent systems of differential equations, there is need to define the adjoint of the system (3.4)–(3.7), i.e.,

$$M^2 f^+_{1R} - 2\alpha_x U\lambda_o M f^+_{1I} = 0 \tag{3.8}$$

$$M^2 f^+_{1I} + 2\alpha_x U\lambda_o M f^+_{1R} = 0 \tag{3.9}$$

$$M g^+_{0R} - 2\alpha_x U\lambda\sigma g^+_{0I} - 2\lambda_o\alpha^2 f^+_{1R} = 0 \tag{3.10}$$

$$M g^+_{0I} + 2\alpha_x U\lambda\sigma g^+_{0R} - 2\lambda_o\alpha^2 f^+_{1I} = 0 \tag{3.11}$$

with the boundary conditions $f_{11} = g_{10} = 0$ at $z = \pm 1/2$.

3.2. $O(\epsilon^2)$ SYSTEM

By separating w_{12} and θ_{11} in real and imaginary parts

$$\theta_{11} = \theta_{1R} + i\theta_{1I}, \qquad w_{12} = w_{2R} + iw_{2I} \tag{3.12}$$

it yields a system of four inhomogeneous partial differential equations

$$\begin{aligned} &M^2 w_{2R} + 2\lambda_0\alpha_x U M w_{2I} - 2\lambda_0\alpha^2\theta_{1R} = \\ &= 2\lambda_0\frac{\partial B}{\partial\tau} M f_{1R} - 2\alpha_x U\lambda_1 B M f_{1I} \\ &+ 2\lambda_1\alpha^2 g_{0R} B + 4(\alpha_x\frac{\partial B}{\partial X} + \alpha_y\frac{\partial B}{\partial Y}). \\ &\cdot[-\lambda_o(g_{0I} + \alpha_x U f_{1R}) + M f_{1I}] \end{aligned} \tag{3.13}$$

$$M^2 w_{2I} - 2\lambda_0 \alpha_x U M w_{2R} - 2\lambda_0 \alpha^2 \theta_{1I} =$$
$$= 2\lambda_0 \frac{\partial B}{\partial \tau} M f_{1I} - 2\alpha_x U \lambda_1 B M f_{1R}$$
$$+ 2\lambda_1 \alpha^2 g_{0I} B + 4(\alpha_x \frac{\partial B}{\partial X} + \alpha_y \frac{\partial B}{\partial Y}) \cdot$$
$$\cdot [\lambda_o (g_{0R} - \alpha_x U f_{1I}) + M f_{1R}] \tag{3.14}$$

$$M\theta_{1R} - 2\alpha_x U \sigma \lambda_0 \theta_{1I} = -2\alpha_x U \lambda_1 \sigma B g_{0I}$$
$$+2(\alpha_x \frac{\partial B}{\partial X} + \alpha_y \frac{\partial B}{\partial Y}) g_{0I} + 2\lambda_0 \sigma \frac{\partial B}{\partial \tau} g_{0R} \tag{3.15}$$

$$M\theta_{1I} - 2\alpha_x U \sigma \lambda_0 \theta_{1R} = 2\alpha_x U \lambda_1 \sigma B g_{0R}$$
$$-2(\alpha_x \frac{\partial B}{\partial X} + \alpha_y \frac{\partial B}{\partial Y}) g_{0R} + 2\lambda_0 \sigma \frac{\partial B}{\partial \tau} g_{0I} \tag{3.16}$$

the boundary conditions being $\theta_{11} = w_{12} = 0$ at $z = \pm 1/2$.

It can be seen that the left hand side has the same form as the $O(\epsilon)$ equations. A certain validity condition involving the adjoint has to be satisfied. By multiplying the inhomogeneous differential system with the adjoint $f_{1R}^+, f_{1I}^+, g_{0R}^+$ and g_{0I}^+, respectively, adding and integrating over the domain of variance of z, it leads to a differential equation for the amplitude function $B(X,Y,\tau)$.

$$\frac{\partial B}{\partial X} + \frac{\alpha_y}{\alpha_x} \frac{\partial B}{\partial Y} + \frac{1}{\bar{c}} \frac{\partial B}{\partial \tau} = \frac{U \gamma \lambda_1}{\gamma_1} B \tag{3.17}$$

where,

$$\gamma_1 = 2 \int_{-1/2}^{1/2} \{2[M f_{1I} - \lambda_o (g_{0I} + \alpha_x U f_{1R})] f_{1R}^+ -$$
$$-2[M f_{1R} - \lambda_o (g_{0R} - \alpha_x U f_{1I})] f_{1I}^+ + (g_{0I} g_{0R}^+ + g_{0R} g_{0I}^+)\} \mathrm{d}z \tag{3.18}$$

$$\gamma_2 = 2 \int_{-1/2}^{1/2} \left[f_{1R}^+ M f_{1R} + f_{1I}^+ M F_{1I} + \sigma (g_{0R} g_{0R}^+ + g_{0I} g_{0I}^+) \right] \mathrm{d}z \tag{3.19}$$

$$\gamma = 2 \int_{-1/2}^{1/2} \left[f_{1I}^+ M f_{1R} - f_{1R}^+ M f_{1I} + \sigma (g_{0R} g_{0I}^+ - g_{0I} g_{0R}^+) \right] \mathrm{d}z \tag{3.20}$$

and $\bar{c} = \gamma_1 \alpha_x / \gamma_2 \lambda_o$ is the group velocity determined with the aid of adjoint functions.

The solution of (3.17) can be described in the following manner:

$$B(X, Y, \tau) = \exp(U \gamma \lambda_1 \tau \bar{c} / \gamma_1) \Psi(X/2 + \alpha_x Y / 2\alpha_y - \bar{c}\tau) \tag{3.21}$$

where $\Psi(X/2 + \alpha_x Y/2\alpha_y - \bar{c}\tau)$ is a unit 2π periodic function and defined as follows.

$$\Psi(X/2 + \alpha_x Y/2\alpha_y - c\bar{\tau}) = a_0 +$$
$$+\Sigma_{j=1}^{m} \left[a_j \cos j(X/2 + \alpha_x Y/2\alpha_y - c\bar{\tau}) + \right.$$
$$\left. + b_j \sin j(X/2 + \alpha_x Y/2\alpha_y - c\bar{\tau} \right] . \tag{3.21a}$$

A neutral solution, which is neither growing nor decaying in time or space, is obtained by seeking a solution which is periodic in τ, which implies $\lambda_1 \equiv 0$.

Without loss of generality we may restrict ourselves to a simpler form, as indicated below

$$\begin{aligned}&\Psi(X/2 + \alpha_x Y/2\alpha_y - c\bar{\tau}) = \\ &a_0 + a_m \cos m(X/2 + \alpha_x Y/2\alpha_y - c\bar{\tau}) + \\ &+ b_m \sin m(X/2 + \alpha_x Y/2\alpha_y - c\bar{\tau})\end{aligned} \tag{3.21b}$$

where m has an integer value.

Next, the functions θ_{11} and w_{12} can be expressed as:

$$\theta_{11} = g_{11}(z)\bar{\bar{\Phi}}(X,Y,\tau) + B_1(X,Y,\tau)g_{10}(z) \tag{3.22}$$

$$w_{12} = f_{12}(z)\bar{\bar{\Phi}}(X,Y,\tau) + B_1(X,Y,\tau)f_{11}(z) \tag{3.23}$$

where $\bar{\bar{\Phi}}$ is the known function $\partial B/\partial X$ and the functions $g_{11}(z)$ and $f_{12}(z)$ are governed by ordinary differential equations, as follows:

$$\begin{aligned}M^2 f_{2R} \;&+\; 2\alpha_x U\lambda_o M f_{2I} - 2\lambda_o\alpha^2 g_{1R} = \\ &2M(8\alpha_x f_{1I} - \lambda_o\bar{c} f_{1R}) - 16\alpha_x\lambda_o(\alpha_x U f_{1R} - g_{0I})\end{aligned} \tag{3.24}$$

$$\begin{aligned}M^2 f_{2I} \;&-\; 2\alpha_x U\lambda_o M f_{2R} - 2\lambda_o\alpha^2 g_{1I} = \\ &-\; 2M(8\alpha_x f_{1R} + \lambda_o\bar{c} f_{1I}) - 16\alpha_x\lambda_o(g_{0R} + \alpha_x U f_{1I})\end{aligned} \tag{3.25}$$

$$M g_{1R} + 2\alpha_x U\sigma\lambda_o g_{1I} = 2(2\alpha_x g_{0I} + \lambda_o\sigma\bar{c} g_{0R}) \tag{3.26}$$

$$M g_{1I} - 2\alpha_x U\sigma\lambda_o g_{1R} = 2(-2\alpha_x g_{0R} + \lambda_o\sigma\bar{c} g_{0I}) \tag{3.27}$$

where (f_{2R}, f_{2I}) and (g_{1R}, g_{1I}) are the real and imaginary part of $f_{12}(z)$ and $g_{11}(z)$, respectively.

The solution is determined up to an additive multiple of the corresponding homogeneous system and this is illustrated by the inclusion of terms containing $B_1(X,Y,\tau)$.

3.3. $O(\epsilon^3)$ SYSTEM

In order to obtain the change in the critical Reynolds number there is need to extend the formulation to a higher $O(\epsilon^3)$ system. A system of four inhomogeneous differential equations is obtained for the real and imaginary parts of w_{13} and θ_{12}, with the left hand side being identical to that of the $O(\epsilon)$ and $O(\epsilon^2)$ systems. The right hand side is more complicated and is a function of the $O(\epsilon^2)$ correction λ_2 instead of λ_1 appearing in the system (3.13)–(3.16). This system will have a solution provided the known orthogonality condition involving the adjoint will be satisfied. It leads to an inhomogeneous differential equation for the second amplitude $B_1(X,Y,\tau)$ (see Appendix), as follows:

$$\frac{\partial B_1}{\partial X} + \frac{\alpha_y}{\alpha_x}\frac{\partial B_1}{\partial Y} + \frac{1}{\bar{c}}\frac{\partial B_1}{\partial \tau} = l_1 B + l_2 \tag{3.28}$$

with

$$l_1 = \frac{1}{\gamma_1}(U\gamma\lambda_2 - m^2\frac{\gamma_3}{\alpha_x}) \tag{3.29}$$

and

$$l_2 = m^2 \frac{\gamma_3}{\alpha_x \gamma_1} a_o \tag{3.30}$$

and where m has been defined beforehand. The parameters γ_1 and γ are given by (3.18) and (3.20), respectively, and γ_3 is defined in the Appendix.

The complete solution is given below as:

$$\begin{aligned} B_1(X,Y,\tau) &= \bar{c}\tau\{l_1[a_o + a_m \cos m(\frac{X}{2} + \frac{\alpha_x Y}{2\alpha_y} - \bar{c}\tau) + \\ &\quad + b_m \sin m(\frac{X}{2} + \frac{\alpha_x Y}{2\alpha_y} - \bar{c}\tau)] + l_2\} + \\ &\quad + CB(X,Y,\tau) \end{aligned} \tag{3.31}$$

where C is an arbitrary constant connected with the solution of the homogeneous problem. If we seek a neutrally stable solution in time and space we must choose $l_1 = l_2 = 0$ which finally leads to $a_o = 0$ and

$$\lambda_2 = m^2 \frac{\gamma_3}{\alpha_x U \gamma} \tag{3.32}$$

where λ_2 represents the elevation above the critical Reynolds number due to the travelling waves effect.

4. Discussion

For low frequencies (i.e. $\Omega << 1$) there is an indication that the average mean velocity $\bar{\bar{u}}$ is again positive and proportional to U. The open problem when the fluid is bounded from above by a free-surface will probably show similarity for high frequencies but some differences for low ones. Here a situation of resonance can be encountered when the surface waves continuously absorb energy from the flow, leading to an increase in the amplitude up to the stage of a breakdown. For the open problem, a separate analysis would be needed.

In a similar way to the idealized two-dimensional problem, it has been found that in the present three-dimensional configuration a net mean momentum is present, giving the indication that the velocity fluctuations transfer momentum which is balanced by the stresses caused by the mean velocity field. For large values of the frequency the mean flow is proportional to U^{-4} in agreement with Davey's [3] calculations. It may be aligned along an arbitrary direction by imposing a transverse variation. The longitudinal component of the mean momentum flux was found to be in the opposite direction to the travelling thermal field, at least under the conditions that the frequency parameter is sufficiently high. It can be noted that the stability analysis is not restricted only to the case when the depth of the fluid is small in comparison with the wavelength of the thermal field ($\alpha^2 h^2 << 4\pi^2$).

The Froude number based on the upstream velocity and on the submergence depth, is relatively small over most of the parameter range, while the Reynolds number based on the same parameters, ranges from $\lambda = 1$ for $U = O(10^{-3} m/s)$ to $\lambda = 10^6$ for finite values of U and h.

It is found that for the linearized stability theory the critical Reynolds number shows an $O(\epsilon^2)$ correction due to the travelling waves effect. Stability studies for the wavy instability appearing in the Taylor problem show the same type of $O(\epsilon^2)$ correlation for the linearized analysis, but the nonlinear analysis leads to an $O(1)$ correction in the critical Taylor number [8]. The elevation above the critical Reynolds number is generally positive and depends on the mode number. It is inversely proportional to the velocity and longitudinal wavelength. It would be quite essential to continue with a nonlinear study of the travelling wave instability caused by the same moving thermal forcing and the expectation is that it will show a likewise $O(1)$ correction in the Reynolds number.

5. Appendix

5.1. $O(\epsilon^3)$ CALCULATIONS

The $O(\epsilon^3)$ system of differential equations, in complex form, can be written as follows:

$$\begin{aligned} &M(M-2i\lambda_o\alpha_x U)w_{13} - 2\lambda_o\alpha^2\theta_{12} \\ &= -2\lambda_2\alpha^2\theta_{10} + 2\lambda_o\left(\frac{\partial^2}{\partial X^2}+\frac{\partial^2}{\partial Y^2}\right)\theta_{10} + \\ &+ 4i\lambda\left(\alpha_x\frac{\partial}{\partial X}+\alpha_y\frac{\partial}{\partial Y}\right)\theta_{11} + 2i\alpha_x U\lambda_2 M w_{11} + \\ &+ 2\lambda_o M\frac{\partial w_{12}}{\partial \tau} - 4\left(\alpha_x\frac{\partial}{\partial X}+\alpha_y\frac{\partial}{\partial Y}\right)(iM+\lambda_o\alpha_x U)w_{12} - \\ &- 2\left(\frac{\partial^2}{\partial X^2}+\frac{\partial^2}{\partial Y^2}\right)(M-i\lambda_o\alpha_x U)w_{11} \end{aligned} \tag{A.1}$$

$$\begin{aligned} &(M-2i\alpha_x U\sigma\lambda_o)\theta_{12} = 2i\alpha_x U\sigma\lambda_2\theta_{10} \\ &-2\left[i\left(\alpha_x\frac{\partial}{\partial X}+\alpha_y\frac{\partial}{\partial Y}\right) - \lambda_o\sigma\frac{\partial}{\partial\tau}\right]\theta_{11} - \left(\frac{\partial^2}{\partial X^2}+\frac{\partial^2}{\partial Y^2}\right)\theta_{10}. \end{aligned} \tag{A.2}$$

Separating the functions w_{13} and θ_{12} into their real and imaginary parts yields:

$$M^2 w_{3R} + 2\lambda_o\alpha_x U M w_{3I} - 2\lambda_o\alpha^2\theta_{2R} = F_1(X,Y,\tau,\lambda_2,B) \tag{A.3}$$

$$M^2 w_{3I} - 2\lambda_o\alpha_x U M w_{3R} - 2\lambda_o\alpha^2\theta_{2I} = F_2(X,Y,\tau,\lambda_2,B) \tag{A.4}$$

$$M\theta_{2R} + 2\alpha_x U\sigma\lambda_o\theta_{2I} = F_3(X,Y,\tau,\lambda_2,B) \tag{A.5}$$

$$M\theta_{2I} - 2\alpha U_x\sigma\lambda_o\theta_{2R} = F_4(X,Y,\tau,\lambda_2,B) \tag{A.6}$$

where

$$\begin{aligned} F_1 &= 2\lambda_o M f_{1R}\cdot\frac{\partial B_1}{\partial\tau} + 4[Mf_{1I}+\lambda_o(g_{0I}-\alpha_x U f_{1R})] \\ &\cdot(\alpha_x\frac{\partial}{\partial X}+\alpha_y\frac{\partial}{\partial Y})B_1 - 2[\alpha_x U\lambda_2 M f_{1I} - \lambda_2\alpha^2 g_{0R}]B + \\ &+m^2\{\frac{\alpha^2}{2}(\frac{1}{\alpha_x^2}+\frac{1}{\alpha_y^2})[Mf_{1R}+\lambda_o(\alpha_x U f_{1I}+g_{0R})] + \\ &+2M(\lambda_o\bar{c}f_{2R}-2\alpha_x f_{2I}) + 4\alpha\lambda_o(-g_{1I}+\alpha_x U f_{2R})\} \\ &\cdot\phi_1(X,Y,\tau,m) \end{aligned} \tag{A.7}$$

$$F_2 = 2\lambda_o M f_{1I} \cdot \frac{\partial B_1}{\partial \tau} - 4[M f_{1R} + \lambda_o(g_{0R} + \alpha_x U f_{1I})](\alpha_x \frac{\partial}{\partial X} + $$
$$+\alpha_y \frac{\partial}{\partial Y})B_1 + +2\lambda_2 \alpha_x [UMf_{1R} + \alpha g_{0I}]B + m^2\{\frac{\alpha^2}{2}(\frac{1}{\alpha_x^2} + \frac{1}{\alpha_y^2}) \cdot$$
$$\cdot[Mf_{1I} - \lambda_o(\alpha_x U f_{1R} - g_{0I})] + 4\alpha_x \lambda_o(\alpha_x U f_{2I} + g_{1R}) +$$
$$+2M(\lambda_o \bar{c} f_{2I} - 2\alpha_x f_{2R})\}\phi_1(X, Y, \tau, m) \qquad \text{(A.8)}$$

$$F_3 = -2\alpha_x U \sigma \lambda_2 g_{0I} B + 2g_{0I}(\alpha_x \frac{\partial}{\partial X} + \alpha_y \frac{\partial}{\partial Y})B_1 + 2\sigma \lambda_o g_{0R} \frac{\partial B_1}{\partial \tau} +$$
$$+2m^2[-\alpha_x g_{1I} + \frac{\alpha^2}{8}(\frac{1}{\alpha_x^2} + \frac{1}{\alpha_y^2})g_{0R} + \sigma \bar{c} \lambda_o g_{1R}]$$
$$\phi_1(X, Y, \tau, m) \qquad \text{(A.9)}$$

$$F_4 = 2\alpha_x U \sigma \lambda_2 g_{0R} B - 2g_{0R}(\alpha_x \frac{\partial}{\partial X} + \alpha_y \frac{\partial}{\partial Y})B_1 + 2\sigma \lambda_o g_{0I} \frac{\partial B_1}{\partial \tau} +$$
$$+2m^2[\alpha_x g_{1R} + \frac{\alpha^2}{8}(\frac{1}{\alpha_x^2} + \frac{1}{\alpha_y^2})g_{0I} + \sigma \bar{c} \lambda_o g_{1I}]\phi_1(X, Y, \tau, m) \qquad \text{(A.10)}$$

where

$$\phi_1(X, Y, \tau, m) = B(X, Y, \tau, m) - a_o. \qquad \text{(A.11)}$$

Since we wish to have consistency in our calculations as far as the systems (3.4)–(3.7) and (A.3)–(A.6) are concerned, an orthogonality condition involving the adjoint has to be fulfilled, namely:

$$\int_{-1/2}^{1/2} (F_1 f_{1R}^+ + F_2 f_{1I}^+ + F_3 g_{0R}^+ + F_4 g_{0I}^+)\mathrm{d}z = 0. \qquad \text{(A.12)}$$

After some algebraic manipulations it leads to a first-order partial differential equation for B_1 in the variables X, Y and the time τ, as follows:

$$\frac{\partial B_1}{\partial X} + \frac{\alpha_y}{\alpha_x}\frac{\partial B_1}{\partial Y} + \frac{1}{\bar{c}}\frac{\partial B_1}{\partial \tau} = \frac{1}{\gamma_1}\left(U\gamma\lambda_2 - m^2 \frac{\gamma_3}{alpha_x}\right) B + \frac{m^2 a_o \gamma_3}{\gamma_1 \alpha_x} \qquad \text{(A.13)}$$

γ and γ_1 are already defined and γ_3 appearing in the latest $O(\epsilon^3)$ calculations is defined by the algebraic expression:

$$\frac{1}{2}\gamma_3 = \alpha_x(g_{1I} g_{0R}^+ - g_{1R} g_{0I}^+) - \lambda_o \sigma \bar{c}(g_{1R} g_{0R}^+ + g_{1I} g_{0I}^+) +$$
$$+2\alpha_x \lambda_o [g_{1R} f_{1I}^+ - g_{0R} f_{1R}^+ - \alpha_x U(f_{2I} f_{1I}^+ - f_{2R} f_{1R}^+)] +$$
$$+M[2\alpha_x(f_{2I} f_{1R}^+ - f_{2R} f_{1I}^+) - \lambda_o \bar{c}(f_{2R} f_{1R}^+ + f_{2I} f_{1I}^+)] \qquad \text{(A.14)}$$
$$-\frac{\alpha^2}{4}(\frac{1}{\alpha_x^2} + \frac{1}{\alpha_y^2})[M(f_{1R} f_{1R}^+ + f_{1I} f_{1I}^+) + \lambda_o f_{1R}^+(\alpha_x U f_{1I} - g_{0R}) -$$
$$\lambda_o f_{1I}^+(\alpha_x U f_{1R} + g_{0I}) + \frac{1}{2}(g_{0R} g_{0R}^+ + g_{0I} g_{0I}^+)]. \qquad \text{(A.15)}$$

The method of solution of the differential equation (A.13) is based on the method of characteristics and stability consideration enables to calculate the change in the critical Reynolds number, shown already by (3.32).

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